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ON THE NATURE OF TIME, SPACE AND GRAVITATION*

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The influence of space-time models on the nature of gravitational forces is analysed.

The starting points in the theory of gravitation depend on the introduction of a continuous four-dimensional manifold of points, filled by families of world lines of individual freely moving particles K. The particle motions are described quantitatively by means of three- and four-dimensional velocities and accelerations, which are introduced by imposing a pseudo-Riemann metric the theory. The same effects may, in general, be described by different models which operator with similar devices and concepts but are not directly reducible to one another /1, 2/.

It will be shown below that, in general, and in particular in the Special Theory of Relativity (STR), the gravitational field is closely linked with the way in which space and time are modelled, and with the introduction of external fields for the gravitational forces due to the interaction of individual particles with adjacent particles, which possess mass. It is natural and convenient to describe these fields in coordinate systems which accompany the continuously distributed parts of the material (we consider a dust of moving individual particles which conserve their masses and move without collisions).

We shall henceforth use the accompanying Lagrange coordinate systems $\xi^1, \xi^2, \xi^3, \xi^4 = \tau$, in which an element ds of a world line of the family K can always be written canonically in the metric form /3, 4/

$$ds^2 = d\tau^2 + 2g_{\alpha 4}(\xi^\beta, \tau)\, d\xi^\alpha\, d\tau + g_{\alpha\gamma}(\xi^\beta, \tau)\, d\xi^\alpha\, d\xi^\gamma \tag{1}$$

where we assume for simplicity that the velocity of light $c = 1$, while $\alpha, \gamma = 1, 2, 3$ are indices of summation, and $g_{ij}(\xi^\alpha, \tau)$ are the components of the metric tensor; the coordinate τ is the proper global time /5, 6/ in the family K of world lines, with differential $d\tau$ which is uniquely defined at every point of the family K in the Riemann space corresponding to the metric (1).

In accordance with the kinematic definition of the four-dimensional velocity vector $\mathbf{u} = d\mathbf{r}/d\tau$ and the acceleration vector $\mathbf{a} = d\mathbf{u}/d\tau$, the following relations hold for their components u^i and a^i in any accompanying coordinate systems on world lines of family K, taken in any Riemann spaces (for simplicity we take $c = 1$):

$$u^\alpha = 0,\ u^4 = 1,\ u_\alpha = g_{\alpha 4},\ u_4 = 1 \tag{2}$$

$$a^i_{\text{abs}} = du^i/d\tau + u^j u^k \Gamma^i_{jk} = g^{ij} \frac{\partial g_{j4}}{d\tau} \tag{3}$$

Hence

$$a_\alpha = \partial g_{\alpha 4}(\xi^\gamma, \tau)/\partial\tau = \partial u_\alpha(\xi^\gamma, \tau)/\partial\tau \tag{4}$$

and $a_4 = 0$ since $g_{44} = 1$. In (2)-(4) the world lines and space can be arbitrary, but the coordinates must be accompanying.

It must be said that, in the accompanying coordinate systems, the three-dimensional velocities are zero, though the absolute four-dimensional acceleration $\mathbf{a}_{\text{abs}}$, equal to the corresponding three-dimensional acceleration, is non-zero, since the accompanying system is movable with respect to the locally inertial system and, in general, is deformable.

We know that, in any observer's coordinate system, the components of the vectors $\mathbf{u}, \mathbf{a}_{\text{abs}}$, and of the metric tensor g_{ij} on the corresponding world lines of family K are calculated from the data (1)-(4) by suitable relations for transforming the coordinates $x^i(\xi^\alpha, \tau)$ for the law of motion of the medium from the accompanying system ξ^α, τ to the system x^i of the observer.

The main geometric and physical laws and the defining quantities are best stated and described in the proper and accompanying coordinate systems connected directly with the objects being studied with suitable characterization of events calculated on local inertial reference systems. From the point of view of outside observers, the theory greatly depends on the properties of the processes in the observer's own system.

There are problems, however, the significance of which lies in finding the physical and mathematical effects of different kinds which are in essence concerned with describing the

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effects from the point of view of a fixed observer.

For instance, it is postulated in quantum mechanics that the results cannot in essence be separated from the observer. The same can sometimes be said of relativity theory, in which the results of observer's measurements can seriously depend on the mechanical states of the object and observer. In particular, we are thinking of problems concerned with the typical properties of processes, defined by a complete set of data which are, in general, obtained in non-holonomic locally defined proper systems of reference. A typical example is the global influence of perturbations on the observed orbits of stellar bodies under different kinds of model assumptions.

We can also mention that the readings of inertial instruments mounted on moving objects correspond to the metric in the accompanying reference system.

At the same time, the coordinate τ of global proper time appears in the canonical form of (1), linked with the family K of world lines.

As our basic postulate corresponding to experience, we assume that, in both Newtonian and relativistic mechanics, for every particle with constant mass which carries an infinitesimal acceleration gauge in free flight, there is in vacuo a state of weightlessness which shows itself by the gauge indicator remaining undisturbed.

This means that, given any fixed model of space and time, whether in Newtonian or relativistic theories, in the absence of any external forces other than gravitational, we must have the equations for any particle of constant mass m in its free motion:

$$-m\mathbf{a}_{\text{abs}}+m\mathbf{g}=0, \quad \text{or} \quad \mathbf{a}_{\text{abs}}=\mathbf{g} \tag{5}$$

Here, $\mathbf{g}$ is the acceleration vector of the gravitational force, regarded as a functional of points of space-time and the law of motion, or, in the case of a specifically stated problem for the motion of test particles, as a function of the coordinates of points of space-time.

It must be noted and emphasized that the presence of acceleration $\mathbf{g}$ does not destroy the conditions (5) of weightlessness, and we can regard the inertia force as a like-acting external force of reaction of space-time, introduced as an external connection defined by the metric.

Eqs.(5) hold for any freely moving individual small particle, and are the general equations of celestial mechanics.

To sum up, in our theory below, the modelling of the motion of individual material particles with constant mass, or the motion of dust as a continuum, is linked with the introduction of the accompanying metric (1) for the family K of world lines for four-dimensional space-time, and of the, in general, variable vector of accelerations g as a function of points of space, linked with the presence of gravitational forces in the basic postulate (5).

In Newtonian mechanics the field of accelerations $\mathbf{g}$ is introduced and justified by Newton's law of universal gravitation. This justification is not in general physically acceptable, however, since it contradicts the invariance of gravitational effects under Lorentz transformations and involves action at a distance and the instantaneous propagation of disturbances.

If, however, the field of accelerations $\mathbf{g}$ is taken in accordance with Newton's law of universal gravitation, it can be calculated approximately in practice. On the other hand, experiments show that the laws of motion of celestial bodies, calculated in accordance with Newton's theory, are, in general, in very good agreement with actual observations. This is because, in all possible routine applications, the ratio v^2/c^2 (where $\mathbf{v}$ is the three-dimensional vector equal to the difference between the four-dimensional velocities of the moving particles, and c is the velocity of light), is negligibly small. In this connection, after replacing v^2/c^2 by zero and passing from the proper coordinate system of the particles to the observer's coordinates by means of a Lorentz transformation, the main relativistic effects are eliminated. The local errors generated by this physical incorrectness are extremely small. If intervals appear on the particle world lines in the Riemann spaces, on which v^2/c^2 cannot be assumed to be zero, we can pose the problem of finding the refined field of acceleration in the gravitational field in the context of relativistic gravitation theory.

In this connection, instead of Newton's law of universal gravitation, we have to use extra assumptions that can be checked experimentally.

An example is the assumption used in the General Theory of Relativity (GTR) that $\mathbf{g}=0$, i.e., that gravitational forces are not present when the space has suitable curvature, so that we find in the GTR, in accordance with postulate (5) and relation (4), that $\mathbf{a}_{\text{abs}}=0$ at points of any world lines of test particles. In the GTR, therefore, all the world lines of freely moving particles must be geodesics. Given an arbitrary but fixed pseudo-Riemann space, the equation $\mathbf{g}=0$ clearly contradicts experience.

Obviously, in the STR and other fixed spaces, a field of gravitational acceleration $\mathbf{g}\neq 0$ and a gravitation force $\mathbf{G}=m\mathbf{g}\neq 0$ must certainly be introduced. Nevertheless, for the curved pseudo-Riemann spaces specially defined in the GTR, the condition $\mathbf{g}=0$ may be admissible in certain very important model statements for the appropriate particle motions in corresponding large scale volumes of space and time. This means that mass gravitational effects can be replaced by suitable space curvature, and absolute time can be replaced by the value of the global proper time in (1) for the corresponding family K.

In general, however, we cannot ignore direct experience that reveals the presence of gravitational forces. The best example is Newton's observation of the falling apple. It is also obvious that, in accordance with the law of universal mass gravitation, the weight force **G** in Newtonian mechanics is present, is generated by the acceleration **g**, and is an experimentally established fact.

On the other hand, we know that the equation of field theory in the GTR for dust with density ρ is, in the accepted notation, in any system of coordinates,

$$R^{ij} - {}^1\!/_2 g^{ij} R = \varkappa \rho u^i u^j \tag{6}$$

where $\varkappa$ is a scalar constant coefficient. On the basis of Bianchi's identity, we obtain from (6): $u^k \nabla_i \rho u^i + \rho u^i \nabla_i u^k = 0$. Hence it follows that $\nabla_i (\rho u^i) = 0$ since the particle masses are constant, while, by (5), the second term gives

$$u^i \nabla_i u^k = du^k / d\tau = g^k = 0$$

It therefore follows from (6) that $g = 0$ at any regular points of space for any finite values of ρ.

In general, the field of the vector **g** or the field of its gradients can be measured in experiments by special instruments /7/, and this can serve as an experimental justification for its determination, which may not in fact be in agreement with Newton's law of universal gravitation or with the GTR for physical objects which possess mass.

It should be said that we are speaking of the replacement of the law of universal mass gravitation in Newtonian mechanics by relativistic laws of interaction between individual particles; the difference between these laws is due to the high three-dimensional velocities of the particles and the four-dimensional curvature of space.

When, instead of Newton's model for space-time, we choose any fixed four-dimensional pseudo-Riemann space, e.g., Minkovskii space in the STR or in general any other specific Riemann space, we can pose the problem of finding the corresponding properties of the acceleration **g** field or the gravity force **G** by analogy with Newtonian physics, e.g., by conversion of the experimentally tested field of three-dimensional accelerations in Newtonian mechanics, which are found by means of universal graviation or by direct measurement of the accelerations or their gradients in experiments with freely moving particles.

Let us mention a method whereby, knowing the field of accelerations $\mathbf{g}^* \neq 0$ in some fixed pseudo-Riemann space R^*, we can find by conversion the field accelerations **g** for the similar problem of the motion of a material medium from the point of view of any observer in the fixed space R^* or in another fixed pseudo-Riemann space $R \neq R^*$. (Since the data of the vector **g** must strictly speaking always be checked by experiment, extra hypotheses are required to specify the field **g** with $v^2/c^2 \approx 1$).

By the analogy of the problems we mean that the accompanying systems of coordinates ξ^1, ξ^2, ξ^3 and the family of lines K are the same, but the proper global times and the metric components g_{ij} may be different for the same world lines $\xi^\alpha = \xi_0{}^\alpha = \text{const}$, which fill the entire space.

Note that, in this case, in each of the spaces R^* and R the metric tensors $g_{ij}{}^*$ and g_{ij} in the accompanying coordinates can be reduced to the canonical form (1), whence it follows that, generally speaking, $d\tau^* \neq d\tau$ if $R^* \neq R$.

If $g_{ij}{}^* = g_{ij}$, then $R^* = R$; in this case, however, along with the system of accompanying coordinates ξ^α, τ we can also introduce the system of observer's coordinates x^α, t with $x^\alpha \neq \xi^\alpha$ and $dt \neq d\tau$; but nevertheless we can find the functions

$$x^k = x^k(\xi^\alpha, \xi^4 = \tau), \; k = 1, 2, 3, 4 \tag{7}$$

where in accordance with (7), from the invariance of the metric form ds^2 in the case $R^* = R$, it is found that

$$g_{ij}{}^*(\xi^k) \neq g_{ij}(x^k), \quad \text{where } g_{ij}{}^*(\xi) = g_{pq}(x) \frac{\partial x^p}{\partial \xi^i} \frac{\partial x^q}{\partial \xi^j} \tag{8}$$

When $R^* = R$ the determination of the four functions (7), or alternatively, the law of motion of points of the continuum on passing from variables ξ^k to variables x^k, is a problem of inertial navigation. We know that it can be solved experimentally, or by theoretical calculations for the reference systems ξ^k and x^k of (7), when $R^* = R$ are given /4/. In this case, relations (7) are generalized Lorentz transformations. Obviously, the transition from one solution to another with fixed space R^* amounts to the transformation at each point of space of the vector **g** to the vector **g*** for a fixed metric, in accordance with the tensor expression (8).

If, in the same Riemann space, an exact solution of the problem of the motion of dust in variables $x^\alpha, x^4 = t$ as a function of the Lagrange variables $\eta^\alpha, \tau = \eta^4$ is obtained, then this defines its own four-dimensional transformation (7) as

$$x^\alpha = \varphi^\alpha(\eta^\beta, t), \quad x^4 = t \tag{9}$$

Here, obviously, relations (9) define the law of motion $x^i(\eta^k)$ in a metric in the same space R, while the three-dimensional components of the velocity v^α can also be regarded, in view of (9), as functions of the accompanying coordinates η^α and t, which will not in general be canonical in the sense of definition (1).

If the variables x^i are taken in the same Cartesian coordinates of Newton and Minkovskii in the STR, then obviously, observers can be introduced into the Minkovskii and Newton spaces, for which the description of motions is well defined by the same functions

$$x^\alpha = \varphi^\alpha(\eta^\beta, t) \tag{10}$$

where t is Newtonian absolute time, which is not, however, equal to Minkovskii proper time τ, while the Lagrange coordinates $\eta^\alpha = \text{const}$ define the same specific law of motion in both cases for different observers. When solving a problem on the motion of particles, we can use Newton's metric

$$dl^2 = dx^{1^2} + dx^{2^2} + dx^{3^2},\ dt$$

or Minkovskii's metric

$$ds^2 = dt^2 - dx^{1^2} - dx^{2^2} - dx^{3^2} = dt^2 - dl^2$$

and the law of motion (10).

Our future arguments are based on transformation (10), which can, in general, be obtained by means of different statements of problems of dust motion.

On the basis of the coordinate transformation (9), which is not a Lorentz transformation with passage from x^i to η^k, we can write

$$\frac{\partial t}{\partial t} = 1, \quad \frac{\partial t}{\partial \eta^\alpha} = 0, \quad \frac{\partial x^\alpha}{\partial t} = v^\alpha$$

$$\hat{g}_{44} = g_{pq}\frac{\partial x^p}{\partial t}\frac{\partial x^q}{\partial t} = 1 - v^2$$

$$\hat{g}_{\alpha 4} = g_{pq}\frac{\partial x^p}{\partial \eta^\alpha}\frac{\partial x^q}{\partial t} = -\frac{\partial x^1}{\partial \eta^\alpha}\frac{\partial x^1}{\partial t} - \frac{\partial x^2}{\partial \eta^\alpha}\frac{\partial x^2}{\partial t} - \frac{\partial x^3}{\partial \eta^\alpha}\frac{\partial x^3}{\partial t}$$

$$\hat{g}_{\alpha\beta} = g_{pq}\frac{\partial x^p}{\partial \eta^\alpha}\frac{\partial x^q}{\partial \eta^\beta} = -\frac{\partial x^1}{\partial \eta^\alpha}\frac{\partial x^1}{\partial \eta^\beta} - \frac{\partial x^2}{\partial \eta^\alpha}\frac{\partial x^2}{\partial \eta^\beta} - \frac{\partial x^3}{\partial \eta^\alpha}\frac{\partial x^3}{\partial \eta^\beta}$$

and hence

$$ds^2 = (1 - v^2)(dt)^2 + 2(\mathbf{e}_\alpha \cdot \mathbf{v})\, d\eta^\alpha dt + (\mathbf{e}_\alpha \mathbf{e}_\beta)\, d\eta^\alpha d\eta^\beta \tag{11}$$

where $\mathbf{e}_\alpha$ are covariant components of the coordinate bases in system η^k, and dt is the differential of Newtonian absolute time, the same as observer's time in the STR. Relation (11) gives the metric in the accompanying system of η^α, t coordinates in Minkovskii space. To obtain the metric in the accompanying canonical system, it suffices to make the further four-dimensional transformation

$$\xi^\alpha = \eta^\alpha, \ d\tau = dt\sqrt{1 - v^2}$$

after which we obtain the accompanying metric in the canonical form

$$ds^2 = d\tau^2 + 2g_{\alpha 4}d\xi^\alpha d\tau + g_{\alpha\beta}d\xi^\alpha d\xi^\beta$$

where $g_{\alpha 4} = v_\alpha/\sqrt{1 - v^2} = u_\alpha$, so that (4) is satisfied, while $d\tau$ is an element of global time.

If in general space $R^* \neq R$, we can also introduce the same family K of world lines, the transformation (7), and different tensor components of metrics g_{ij}^* in R^* and g_{ij} in R, but the connection (8) between these metric tensors is destroyed.

Accordingly, along the same lines K we can define separately the local kinematic tensor characteristics, e.g., for the four- and three-dimensional velocities $\mathbf{u}$ and $\mathbf{v}$, and for acceleration $\mathbf{a}_{abs}$ in R^* by means of g_{ij}^*, and in R by means of g_{ij}, on the basis of canonical formula (1).

It is also obvious that the different vectors and relations between them, defined in R^*, and the similarly defined analogous vectors in the same sense in R, can also be considered in R^*, but with a suitably changed meaning.

This situation means that we can pass from relations and characteristics in one Riemann space to similar relations and characteristics in another Riemann space. In particular, we can introduce $\mathbf{a}^*_{abs}$ into R^* and $\mathbf{a}_{abs}$ into R along the same lines $\xi^\alpha = \text{const}$. Hence, obviously, in accordance with the weightlessness conditions (5), we find that, along with the equations $\mathbf{a}^*_{abs} = \mathbf{g}^*$ in R^*, the equations $\mathbf{a}_{abs} = \mathbf{g}$ in R must also be satisfied, while it turns out that

$g^* \neq g$ and $a^*_{abs} \neq a_{abs}$.

As particular examples of different canonical metrics (1) with $K = K'$ but $R^* \neq R$, we can take

$$ds^2 = d\tau^2 + 2kg_{\alpha 4}d\xi^\alpha d\tau + g_{\alpha\beta}d\xi^\alpha d\xi^\beta$$

where $k > 0$ is a constant scalar. It is easily seen that, in this case, for the acceleration components along the wrold lines K we have $a_\alpha' = ka_\alpha$.

Thus, from the acceleration field in R^* we find the acceleration field in R. For instance, from the acceleration field in the STR we can find the acceleration field in any model Riemann space, i.e., in Riemann spaces with $K = K'$, different metrics and components of the acceleration a_α and a_α' are obtained.

It should be emphasized that the components of the acceleration field a_α in a fixed space with $R^* = R$ are transformed for different $K\,(\eta^\alpha \to \xi^\alpha)$ as components of the same three-dimensional vectors $a = g$, while on transition from one space to another the vectors are changed while keeping them equal as characteristics of the space itself.

For the same family of world lines K we can consider different Riemann spaces R^* and R and accelerations g^* and g, if the transformations (7) are chosen or designated.

On the other hand, if the space R^* is fixed, then at every point of it we can use inertial navigation to find relations (7) for different observers, and the corresponding families K and K', along with the corresponding components of the accleration vector for the same vector g.

In particular, if the metric for the family $\xi^\alpha = \xi_0^\alpha = \text{const}$ in the accompanying coordinate system has the form

$$ds^2 = d\tau^2 + 2g_{\alpha 4}(\xi^\beta)\, d\xi^\alpha\, d\tau + g_{\alpha\gamma}(\xi^\beta, \tau)\, d\xi^\alpha\, d\xi^\gamma \tag{12}$$

where τ is proper time, then at any point of these spaces we find that $a = g = 0$. Versions of the metric are obtained, corresponding to the general theory of relativity, i.e., the presence of the field g is replaced by curvature of space, in which all the world lines of test particles are geodesics.

Since, under any coordinate transformations in a fixed space, geodesics always transform into geodesics, the general forms of metrics (1) and (12) are obviously invariant.

Notice also that, given any pseudo-Riemann space, the vector a_{abs} can easily always be calculated in the accompanying coordinate system (1), and hence, on the basis of (5), the field for the vector g can be obtained, provided that the field is known in a given space, e.g., in the STR.

It is obviously not possible, merely from the weightlessness conditions (5), and from the condition represented by (5) that external forces of a non-gravitational type be absent in free motion, to establish the metric (1) uniquely; this is bound up with the need to choose a model of relativistic space-time, and also an acceleration field for the gravity forces g. In this connection we can pose problems of choosing auxiliary assumptions for establishing the model metrics of the medium of pseudo-Riemann four-dimensional spaces.

In particular, it is easily seen that there is a vast class of pseudo-Riemann spaces in which motions of the dust matter can occur only under the conditions $a_{abs} = g = 0$, which appear in the general theory of relativity.

To choose specific model spaces, apart from specifying an experimentally justified gravitational field of accelerations $g \neq 0$ or $a_{abs} \neq 0$, we also need a number of basic additional assumptions, which bring in physically justified functionals and variational methods for obtaining the appropriate equations (they may be equations of the Hilbert-Einstein type; depending on the different specific problems, the Riemann spaces that result from solving them can differ widely).

This situation makes the theory much more complicated. The model theory of Newtonian graviation in fixed space is obviously simple and describes the effect of gravitation with very great accuracy. In this context, Logunov's idea of constructing a similar mechanical theory in the STR must be recognized as natural. Our above theoretical results make clear the physical essence of gravitation and the ways of constructing new models that take account of relativistic effects, if we recall that all the mathematical models in physics can be coarsened and improved, and there are good reasons for this in the GTR.

At the same time, in present-day practice, given suitable equations in the GRT, unique solutions on dust motions can only be obtained, strictly speaking, in particlar examples and only with supplementary particular assumptions. The standardization of statements of problems in the GTR for partial differential equations by means of conditions of the initial or boundary value type has not yet been introduced in applications /8, 9/.

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ON COMPLICATED MODELS OF CONTINUOUS MEDIA IN THE GENERAL THEORY OF RELATIVITY*

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In the context of the general theory of relativity, the system of Euler's equations is obtained from the variational equation under the assumption that the Lagrangian of the material depends on supplementary (as compared with classical theories) thermodynamic parameters, and when possible irreversible processes are taken into account. It is shown that, for a thermodynamically closed system, the equations of momenta for a continuous medium are a consequence of the field equations. The form of the energy-momentum tensor of the material is considered when the arguments include the Lagrangian of the derivatives of the supplementary thermodynamic parameters.

Let x^i be the coordinates in four-dimensional Riemann space, in which the components of the metric tensor g_{ij}, the coefficients of parallel transfer $\Gamma_{ij}{}^k$, and the curvature tensor $R_{ijk}^{\cdot\cdot\cdot s}$ are connected by the equations

$$ds^2 = g_{ij}dx^i dx^j \tag{1}$$

$$\Gamma_{ij}{}^k = \frac{1}{2} g^{ks}\left[\frac{\partial g_{ks}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right]$$

$$R_{ijk\cdot}^{\cdot\cdot\cdot s} = \frac{\partial \Gamma_{ik}^s}{\partial x^j} - \frac{\partial \Gamma_{jk}^s}{\partial x^i} + \Gamma_{pj}^s \Gamma_{ki}^p - \Gamma_{pi}^s \Gamma_{kj}^p$$

$$R_{ij} = R_{isj\cdot}^{\cdot\cdot s}, \quad R = R_{ij} g^{ij}$$

where R_{ij} are the components of the Ricci tensor, and R is the scalar curvature of the space (the Ricci scalar). Throughout, the small Latin indices cover the values 1, 2, 3, 4; summation is performed with respect to repeated sub- and super-scripts; the signature of the metric is $(+ - - -)$.

Together with the variables x^i in the Riemann space we consider for a solution the accompanying coordinates ξ^k, in which fixed values ξ^1, ξ^2, ξ^3 individualize a point of the continuous medium; we assume that there is a one-to-one correspondence $x^i = x^i(\xi^k)$ between the variables x^i and ξ^k, which is the law of motion of the continuum of the continuous medium.

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